Explicit Construction of Certain Positive Homogeneous ReLU Networks

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Let $F : \mathbb{R}^n \to \mathbb{R}$ be a Minkowski norm (smooth away from the origin, positive homogeneous of degree one). Consider the map

$$J(\xi) = \nabla \left\{ \frac{1}{2} F^2(\xi) \right\},$$

noting that J is also positive homogeneous of degree one. We wish to parameterize J as a shallow neural network using a ReLU activation. Since J is the gradient of a function, its Jacobian must be symmetric (since the Jacobian of J is the Hessian of a smooth scalar-valued function). This constrains our neural network representation to be of the form

$$\widehat{J}(\xi) = W^\top \sigma(W\xi),$$

where W is a matrix, and σ denotes the elementwise ReLU activation function. Taking the limit of infinite width, we can write this as an integral on a measure on $\mathbb{R}^n$:

$$\widehat{J}(\xi) = \int_{\mathbb{R}^n} w \cdot \sigma(\langle w, \xi \rangle) d\mu(w),$$

where μ is a finite measure on $\mathbb{R}^n$. By positive homogeneity of σ, this can be reduced to an integral over the sphere:

$$\widehat{J}(\xi) = \int_{S^{n-1}} w \cdot \sigma(\langle w, \xi \rangle) d\nu(w).$$

Since $\widehat{J}$ is a vector field given by the gradient of a scalar function, we apply the lower Hodge Laplacian to yield a convenient formula. For $\|\xi\| = 1$, we have

$$\nabla (\mathrm{div} J)(\xi) = \int_{S_\xi} w d\nu(w),$$

where S_ξ denotes the great circle of unit vectors perpendicular to ξ. This is nothing more than the Funk transform (or spherical Radon transform) of the signed vector-valued measure $w \cdot \nu$. The Funk transform can only recover the even part of its integrand. Observe that w is an odd function on the sphere Therefore:

Lemma 1. *The odd part of the measure ν is determined by J.*

We denote the part of the function J determined by the odd part of ν by J_o, so that

$$J_o(\xi) = \int_{S^{n-1}} w \cdot \sigma(\langle w, \xi \rangle) d\nu_o(w).$$

The even part, fortunately, reduces to a very simple form. To see this, consider an even measure ν_e on the sphere S^{n-1}. See that the function determined by ν_e is linear:

$$\begin{aligned}
J_e(\xi) &= \int_{S^{n-1}} w \cdot \sigma(\langle w, \xi \rangle) d\nu_e(w) \\
&= \int_{S^{n-1}_+} w \cdot \sigma(\langle w, \xi \rangle) - w \cdot \sigma(\langle -w, \xi \rangle) d\nu_e(w) \\
&= \int_{S^{n-1}_+} w \cdot \left(\sigma(\langle w, \xi \rangle) - \sigma(\langle -w, \xi \rangle)\right) d\nu_e(w) \\
&= \int_{S^{n-1}_+} w \cdot \langle w, \xi \rangle d\nu_e(w) \\
&= \frac{1}{2} \int_{S^{n-1}} w w^\top d\nu_e(w) \xi \\
&= \frac{1}{2} \Sigma_\nu \xi,
\end{aligned}$$

where S^{n-1}_+ denotes a hemisphere. Succinctly,

Lemma 2. *The even part of the measure ν is determined by J only up to its second moment Σ_ν, which in turn determines J_e.*

By this lemma, we can safely denote $\Sigma_J = \Sigma_\nu$ without ambiguity. We see then that $J(\xi) = J_o(\xi) + \frac{1}{2}\Sigma_\nu \xi$. Since the construction of J is such that it is the gradient of a squared norm, we can recover the norm by Euler's homogeneous function theorem:

$$F^2(\xi) = 2\langle \xi, J(\xi) \rangle = 2\langle \xi, J_o(\xi) \rangle + \langle \xi, \Sigma_J \xi \rangle.$$