Gaussian white noise on Hilbert spaces

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1 Introduction

White noise is basically a hack, and I am surprised that we so casually claim that it exists. The following sentence is amazingly common:

> Let x be an element of a Hilbert space H, and consider observations of the form $y = Ax + \eta$, where $A : H \to H$ is a linear operator and η is white gaussian noise.

I would like to focus on the statement "η is white gaussian onise." What even is that?

Let H be a separable, infinite-dimensional Hilbert space. We would like to define a notion of "white gaussian noise" for this space.

2 First attempt: Direct definition

Define a probability measure μ on H such that for any orthonormal basis $\{\phi_j\}_{j\geq 1}$ for H, we have for $f \sim \mu$,

$$\langle f, \phi_j \rangle \sim \mathcal{N}(0,1) \; \forall j \geq 1, \tag{1}$$

and for $j \neq k$, $\langle f, \phi_j \rangle$ and $\langle f, \phi_k \rangle$ are independent. In fact, this independence will hold for any pair of orthogonal vectors.

This is problematic: clearly, $f \sim \mu$ almost surely has infinite norm. In some sense, H being a Hilbert space demands a regularity/decay property of all of its elements, which is too strong for the existence of something like white noise. Based on the ideas for formalizing distributions, such as the Dirac delta function, we will try to cast white noise as some measure on the dual of a sufficiently regular space. In many cases, making a space smaller (in this case, more regular) means that its dual becomes larger (in this case, less regular).

3 Second attempt: Regards to Sobolev

One concept of white noise is understood as something whose basis transformations yield i.i.d. sequences of normal random variables. Let us pick a fixed basis $\{\phi_j\}_{j\geq 1}$ for H. The inner product on H is then written, for $u, v \in H$,

$$\langle u, v \rangle = \sum_{j \geq 1} \langle u, \phi_j \rangle \cdot \langle \phi_j, v \rangle. \tag{2}$$

We desire some random $f \in H$ such that

$$\langle f, u \rangle = \sum_{j \geq 1} \langle f, \phi_j \rangle \cdot \langle \phi_j, u \rangle \tag{3}$$

is normally distributed.

As we saw before, the issue comes with the above inner product being finite. Elements $u \in H$ are such that the sequence $(\langle \phi_j, u \rangle)_{j \geq 1}$ is *square summable*, but not necessarily *absolutely summable*. This motivates us to consider a subspace of H with a stronger inner product. Define $H_1 \subset H$ to consist of elements u such that the basis expansion satisfies $|\langle \phi_j, u \rangle| \lesssim j^{-(1+\epsilon)}$, for some arbitrarily small but fixed $\epsilon > 0$. This comes with a stronger inner product on H_1, defined for $u, v \in H_1$ as

$$\langle u, v \rangle_1 = \sum_{j \geq 1} j^{1+\epsilon} \langle u, \phi_j \rangle \langle \phi_j, v \rangle. \tag{4}$$

For $f \in H_1$, we define the sequence $f_j = j^{1+\epsilon} \langle f, \phi_j \rangle$. In fact, we can choose f_j to be any sequence such that $(j^{-(1+\epsilon)} f_j)_{j \geq 1} \in \ell_2(\mathbb{N})$. Then, the inner product on H_1 is equal to

$$\langle f, u \rangle_1 = \sum_{j \geq 1} f_j \langle \phi_j, u \rangle. \tag{5}$$

Define a probability measure μ on H_1 such that for $f \sim \mu$, each f_j is i.i.d. normally distributed. Then, for any $k \geq 1$,

$$\langle f, \phi_k \rangle_1 = \sum_{j \geq 1} f_j \langle \phi_j, \phi_k \rangle = f_j \sim \mathcal{N}(0, 1). \tag{6}$$

Note that each $\phi_k \in H_1$, so this is well-defined. Moreover, we have a normal distribution for *any* $u \in H_1$ such that $\langle u, u \rangle_1 = 1$.

We have seemingly accomplished our goal. However, there is a problem. Suppose we have $u, v \in H_1$ such that u and v are orthogonal with respect to the inner product for H. It is not necessarily the case that they are orthogonal with respect to the inner product for H_1. Because of this, it is also not necessarily true that $\langle f, u \rangle_1$ is independent from $\langle f, v \rangle_1$.

This attempt brings us close, though, as it expresses the idea of "sandwiching" the space H between two spaces: a smaller one (in this case, H_1), and a larger one (the space of sequences f_j). The issue here has arisen from the crude fashion in which the sequence $(f_j)_{j \geq 1}$ is linked to H_1. Our next try will consider a case where H is put between a linear space and its dual without such an identification, preserving the structure of H to get independence of the random functional applied to orthogonal vectors.

4 Third attempt: The game is rigged!

We will employ the idea of a "Gelfand triple." Let Φ be a topological vector space with a continuous inclusion map $\iota : \Phi \hookrightarrow H$ such that $\iota(\Phi)$ is dense in H. Taking the adjoint of this map, we covariantly get an embedding $\iota^* : H^* \hookrightarrow \Phi^*$, where $H \cong H^*$, and Φ^* is the dual of

Φ. We will find it useful to denote the Riesz map $R : H \to H^*$ as the canonical isomorphism between H and H^*.

The pair (H, Φ) is sometimes called a *rigged Hilbert space*; alternatively, the triple (Φ, H, Φ^*) is called a *Gelfand triple*.

The inclusion $\iota^* : H^* \hookrightarrow \Phi^*$ is such that for all $v \in H$ and $u \in \Phi$,

$$\iota^* Rv(u) = \langle v, \iota u \rangle. \tag{7}$$

That is to say, the duality relationship between Φ and Φ^*, the inclusion relationships $\Phi \hookrightarrow H \cong H^* \hookrightarrow \Phi^*$, and the Hilbert space structure are all compatible.

The following theorem will be helpful.

Theorem 1 (Bochner-Minlos)**.** *Let Φ be a nuclear space. Then, $\Psi : \Phi \to \mathbb{C}$ is the characteristic function of a Borel probability measure μ on Φ^* if and only if*

1. $\Psi(0) = 1$,
2. Ψ *is continuous, and*
3. Ψ *is positive definite.*

Suppose our Gelfand triple is such that Φ is a nuclear space. Define $\Psi : \Phi \to \mathbb{C}$ so that for any $u \in \Phi$,

$$\Psi(u) = \exp(-\|\iota u\|^2/2). \tag{8}$$

This clearly fulfills the conditions of the Bochner-Minlos Theorem, indicating that Ψ is the characteristic function of a Borel probability measure μ on Φ^*. That is to say, μ is such that

$$\mathbb{E}_{f\sim\mu}[\exp(if(u))] = \exp(-\|\iota u\|^2/2), \tag{9}$$

which is to say that $f(u) \sim \mathcal{N}(0, \|\iota u\|^2)$ for $f \sim \mu$. We can also verify the independence condition. Let $u_1, u_2 \in \Phi$ be such that $\iota u_1 \perp \iota u_2$ in H. Then, one can check that

$$\Psi(u_1 + u_2) = \Psi(u_1) \cdot \Psi(u_2), \tag{10}$$

which implies $f(u_1)$ and $f(u_2)$ are independent random variables for $f \sim \mu$.

In this sense, we can say that f models white noise using the structure of the Hilbert space H, while being restricted to acting on some smaller space Φ.